Regular Type III and Type N Approximate Solutions

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Abstract

New type III and type N approximate solutions which are regular in the linear approximation are shown to exist. For that, we use complex transformations on self-dual Robinson-Trautman metrics rather then the classical approach. The regularity criterion is the boundedness and vanishing at infinity of a scalar obtained by saturating the Bel-Robinson tensor of the first approximation by a time-like vector which is constant with respect to the zeroth approximation.

1 Introduction

Exact solutions of type III and type N built around twisting congruences are difficult to obtain. Because of this, people have looked at approximate solutions, in principal for the simpler, null case. In this paper we present a method of obtaining a class of type III linear solutions based on complexification of a non-twisting space-time and its complex transformation into a twisting one.

Any real solution of a Lorentz invariant theory can be transformed, via a complex Lorentz transformation, into a complex solution. When dealing with linear solutions such as the results of a linear approximation to a gravitational field, the real and imaginary parts are, in principle, new solutions of the theory [1], [2]. When the additional requirement of preserving the algebraic type of the original electromagnetic field or gravitational field is considered, a complex Lorentz transformation is insufficient. Recall that the algebraic type of the Weyl tensor is a property of the self-dual and anti-self-dual parts taken separately. Algebraic integrity is maintained between the real and complex solutions if one proceeds by performing a complex Lorentz transformation on the self-dual solution and then adding its complex conjugate to obtain the real part. This keeps the algebraic structure of the Weyl tensor and in particular any null bivectors which are coincident remain coincident. At this point note that with the self dual and anti-self-dual solutions defining the principal null congruence, the geometry of the principal null congruence has not been preserved.

We carry out this procedure explicitly starting with a linear approximation of a type III space-time built around the nontwisting Robinson congruence.

After complexifying and putting conditions for self-duality, we apply a complex transformation to obtain the most general expression for type III solutions built around what might be described as a generalized Kerr congruence. A large sub-class of these solutions are shown to be regular using a scalar obtained from their Bel-Robinson tensor.

This method has its limitations. First, one cannot extend it to obtain the most general type III twisting solutions. Second, the obtained solution may not be in a surveyable form .

2 Self-dual Robinson-Trautman spaces

We can complexify the Robinson-Trautman metric, in a purely formal way, by taking the coordinates ρ, σ, ζ, $\widetilde{\zeta}$ to be independent complex variables and the defining functions $m(\sigma)$ and $p(\sigma, \zeta, \widetilde{\zeta})$ to be complex. It is then easy to see that the Weyl tensor is self-dual if, and only if, $m = 0$ and $p^{-1}p_{\widetilde{\zeta}\widetilde{\zeta}}$ is a function of $\widetilde{\zeta}$ only. By means of a coordinates transformation we can strengthen the last condition to $p_{\widetilde{\zeta}\widetilde{\zeta}} = 0$. The line element is then given by

$$ds^2 = 2d\rho d\sigma + \left(K - 2\rho\frac{p_\sigma}{p}\right)d\sigma^2 - \frac{2\rho^2}{p^2}d\zeta d\widetilde{\zeta} \tag{1}$$

$$K = 2(AB_\zeta - A_\zeta B), p = A + \widetilde{\zeta}B \tag{2}$$

where A and B are arbitrary functions of ζ and σ. The Weyl tensor is

$$\begin{aligned} C_{klmn} &= \frac{p}{2\rho^2}K_\zeta(M_{kl}N_{mn} + N_{kl}M_{mn}) - \frac{1}{2\rho^2}(p^2K_\zeta)_\zeta N_{kl}N_{mn} \\ &\quad + \frac{p^2}{\rho}(p^{-1}p_{\zeta\zeta})_\sigma N_{kl}N_{mn} \end{aligned} \tag{3}$$

where

$$N = \frac{\rho}{p}d\sigma \wedge d\zeta, \tag{4}$$

$$M = d\sigma \wedge d\rho - \frac{\rho^2}{p^2}d\zeta \wedge d\overline{\zeta}. \tag{5}$$

3 Change of coordinates

It is worth remarking that, at least in a special case, one can put this line element into the standard form for one with twisting rays. The special case is defined by $p = A(\zeta) + \widetilde{\zeta}B(\zeta)$. The transformation

$$\begin{aligned} \rho &= r - i\Sigma \\ \zeta &= z \\ \widetilde{\zeta} &= \frac{r + ia}{r - ia}\widetilde{z}. \end{aligned} \tag{6}$$

where $\Sigma = a\frac{A(z)-\widetilde{z}B(z)}{P}$ and $P = A(z) + \widetilde{z}B(z)$, a being a real parameter, transforms (1) into

$$ds'^2 = 2\lambda\nu - 2\mu\widetilde{\mu} \tag{7}$$

where

$$\begin{aligned}
\lambda &= d\sigma + 2L dz && (8)\\
\mu &= P^{-1}(r - i\Sigma)dz, && (9)\\
\widetilde{\mu} &= P^{-1}(r + i\Sigma)d\widetilde{z}, && (10)\\
\nu &= dr + i\left(\Sigma_z dz - \Sigma_{\widetilde{z}} d\widetilde{z}\right) + \frac{1}{2}K\lambda. && (11)
\end{aligned}$$

and $L = \frac{ai\widetilde{z}}{P^2}$.

In the new coordinates, the bivectors N and M are given by

$$\begin{aligned}
N_{kl} &= \frac{r - ia}{r - i\Sigma} N'_{kl}\,, && (12)\\
M_{kl} &= M'_{kl} + \frac{2PKL}{r - i\Sigma} N'_{kl}\,, && (13)
\end{aligned}$$

where

$$\begin{aligned}
N' &= \lambda \wedge \mu && (14)\\
M' &= \lambda \wedge \nu - \mu \wedge \widetilde{\mu}. && (15)
\end{aligned}$$

Making this transformation on the flat space line element

$$\begin{aligned}
\underset{0}{ds}^2 &= 2d\rho d\sigma + 2kd\sigma^2 - \frac{2\rho^2}{p^2} d\zeta d\widetilde{\zeta} && (16)\\
\underset{0}{p} &= 1 + k\zeta\widetilde{\zeta}, && (17)
\end{aligned}$$

writing $\overline{z}$ for $\widetilde{z}$ and putting $\sigma = u - ia\frac{z\overline{z}}{\underset{0}{P}}$ with $\underset{0}{P} = 1 + kz\overline{z}$ we get

$$ds^2 = 2\lambda(dr + i\Sigma_z dz - i\Sigma_{\overline{z}} d\overline{z} + k\lambda) - \frac{2(r^2 + \Sigma^2)}{\underset{0}{P}^2} dz d\overline{z} \tag{18}$$

where $\lambda = du + ia\underset{0}{P}^{-2}(\overline{z}dz - zd\overline{z})$ and $\Sigma = a\frac{1-kz\widetilde{z}}{1+kz\widetilde{z}}$. Note that, for $k = 1$, this is the transformation discussed by Newman in [5], where the connection between the different set of coordinates is achieved via

$$\zeta = \tan\frac{\theta}{2}e^{i\varphi}. \tag{19}$$

4 Linear Approximation

We now examine perturbations of the line element (1) by writing

$$p = \underset{0}{p} + \varepsilon \underset{1}{p} \tag{20}$$

and working to an accuracy of the first order in ε. It is convenient to take

$$\underset{1}{p} = \alpha_\zeta + \beta\overline{\zeta} + k\overline{\zeta}(\zeta\alpha_\zeta - 2\alpha), \tag{21}$$

where $\alpha = \alpha(\sigma, \zeta)$ and $\beta = \beta(\sigma, \zeta)$ are arbitrary functions.

The Weyl is given by

$$\underset{0}{C}_{klmn} = 0, \tag{22}$$

$$\begin{aligned} \underset{1}{C}_{klmn} &= \frac{1}{\rho^2}\underset{0}{p}\Phi(M_{kl}N_{mn} + N_{kl}M_{mn}) \\ &\quad - \left[\frac{1}{\rho^2}\left(\underset{0}{p}^2\Phi\right)_\zeta - \frac{1}{\rho}\underset{0}{p}\left(\Psi\underset{0}{p} + \Phi_\sigma\widetilde{\zeta}\right)\right] N_{kl}N_{mn} \end{aligned} \tag{23}$$

where $\Phi = \beta_{\zeta\zeta}$ and $\Psi = \alpha_{\zeta\zeta\zeta\sigma}$.

Next we transform to the coordinates u, r, $z, \overline{z}$, restrict ourselves to the space-time in which u and r are real, z and $\overline{z}$ are complex conjugates and we consider the line element $ds^2 = \underset{0}{ds}^2 + \varepsilon\underset{1}{ds}^2 + \varepsilon\underset{1}{d\overline{s}}^2$. Then, using our transformation of bivectors and our expression for C in the linear approximation we get

$$C_{klmn} = \varepsilon\left[X(M'_{kl}N'_{mn} + N'_{kl}M'_{mn}) + YN'_{kl}N'_{mn}\right] + c.c. \tag{24}$$

where

$$X = \frac{P}{(r - ia)(r - i\Sigma)}\Phi\,, \tag{25}$$

$$Y = -\frac{P}{(r-ia)^2}\left[\frac{2k\overline{z}(r+ia)}{r - i\Sigma}\Phi + P\Phi_z - \Psi P(r - i\Sigma) - \Phi_\sigma(r+ia)\overline{z}\right]. \tag{26}$$

The Bel-Robinson tensor can then be written as

$$\begin{aligned} \frac{1}{2}P_{abcd} &= {}^{+}\underset{1}{C}_{amnc}\,{}^{-}\underset{1}{C}_{b}{}^{mn}{}_{d} \\ &= 4\left|\Gamma\right|^2\left[\lambda_{(a}\lambda_b\lambda_c\nu_{d)} + 3\lambda_{(a}\lambda_b\mu_c\overline{\mu}_{d)}\right] \\ &\quad + 4\Gamma\overline{\Delta}\lambda_{(a}\lambda_b\lambda_c\overline{\mu}_{d)} \\ &\quad + 4\overline{\Gamma}\Delta\lambda_{(a}\lambda_b\lambda_c\mu_{d)} \\ &\quad + \left|\Delta\right|^2\lambda_a\lambda_b\lambda_c\lambda_d \end{aligned} \tag{27}$$

where

$$\Gamma = X\frac{r - ia}{r - i\Sigma}\,, \tag{28}$$

$$\Delta = \frac{2XPKL(r - ia)}{(r - i\Sigma)^2} + Y\left(\frac{r - ia}{r - i\Sigma}\right)^2. \tag{29}$$

5 An auxiliary metric and the gravitational density

There are two ways of measuring the gravitational field: first one can look at the differential invariants obtained from the Weyl tensor; second one looks at the sum of the squares of its components.

We want to apply the second procedure, but we need a positive definite metric for that. We can create one and take the sum of the squares using this auxiliary metric.

Let

$$\gamma_{ab} = 2t_a t_b - t_r t^r g_{ab} \tag{30}$$

where t is a timelike (unit) vector. We have [3]

$${}^{+}C_{abcd}{}^{-}C_{rstu}\gamma^{ar}\gamma^{bs}\gamma^{ct}\gamma^{du} = (t_r t^r)^2 P_{abcd} t^a t^b t^c t^d. \tag{31}$$

This expression usually depends on t and is of no interest, but in the linear approximation we can take t to be constant with respect to the background (Kerr metric in this case). The easiest way to find such a vector field is to go to Cartesian coordinates. This is accomplished by the following transformation (see [4])

$$U = u + \frac{rz\overline{z}}{P} \tag{32}$$

$$V = \frac{r}{P} + ku \tag{33}$$

$$Z = \frac{r - ai}{P} z \tag{34}$$

where the capital letters represent the Minkowski coordinates. Any constant timelike one form, i.e.

$$c_0 dU + c_1 dV + c_2 dZ + \overline{c}_2 d\overline{Z} \tag{35}$$

$c_i = const.(i = 1, 2, 3)$ such that $c_0 c_1 - c_2 \overline{c}_2 > 0$, represents a constant timelike one-form in Kerr coordinates.

6 Discussion

(a) For $k = 0$ a constant tetrad is

$$\widehat{\nu} = \nu \tag{36}$$

$$\widehat{\mu} = \mu + z\nu \tag{37}$$

$$\widehat{\lambda} = \lambda + \overline{z}\mu + z\overline{\mu} + z\overline{z}\nu. \tag{38}$$

A convenient timelike one form is $\tau = \widehat{\lambda} + \widehat{\nu}$; let t be the vector field associated with it. We have

$$\frac{1}{2} P_{abcd} t^a t^b t^c t^d = \left\{ 4 \left| \Gamma \right|^2 + \left| 4\overline{z}\Gamma - (1 + z\overline{z})\Delta \right|^2 \right\} (1 + z\overline{z})^2, \tag{39}$$

and

$$\begin{aligned} \Gamma &= \frac{\Phi}{(r - ia)^2}, &(40)\\ \Delta &= Y|_{k=0} \\ &= -\frac{1}{(r-ia)^2} \left[\Phi_z - \Psi(r - ia) - \Phi_\sigma (r + ia)\overline{z} \right]. &(41) \end{aligned}$$

One can get directional nonsingular and asymptotically flat solutions by setting $\Phi = \frac{1}{(\sigma - i)^n}$ with $n > 1$ and $\Psi = \frac{1}{(\sigma - i)^m}$ with $m > 1$.

(b) If $k > 0$, a constant timelike one form is $\tau = k\lambda + \nu$. We have

$$\frac{1}{2} P_{abcd} t^a t^b t^c t^d = 4k \left| \Gamma \right|^2 + \left| \Delta \right|^2. \tag{42}$$

For $a = 0$ we have

$$\begin{aligned} \Gamma &= \frac{P}{r^2} \Phi, &(43)\\ \Delta &= Y|_{a=0} \\ &= -\frac{P}{r^2} \left[2k\overline{z}\Phi + P\Phi_z - \Psi P r - \Phi_\sigma r \overline{z} \right] &(44) \end{aligned}$$

The situation is similar to the null case, directional singularities being unavoidable.

References

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